L^2-Betti numbers and their applications

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Some motivation

- We start with presenting some interesting and comprehensible results from different areas of mathematics.

Theorem (Euler characteristic of amenable groups, Cheeger-Gromov)

Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite model for BG.

Then its Euler characteristic satisfies

$$\chi(BG) = 0.$$

Definition (Amenable group)

A group G is called amenable if there is a left G-invariant linear operator $\mu\colon L^\infty(G,\mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$ which satisfies for all $f \in L^\infty(G,\mathbb{R})$

$$\inf\{f(g) \mid g \in G\} \le \mu(f) \le \sup\{f(g) \mid g \in G\}.$$

- A group which contains F_2 as a subgroup is never amenable.

- There are non-amenable groups which do not contain F_2 as subgroup.

- But they are hard to construct and a group which does not contain F_2 is very likely to be amenable.

- Solvable groups are amenable.

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\text{defi}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

- The deficiency is an important invariant in group theory and low-dimensional topology.

- Lower bounds can be obtained by investigating specific presentations. The hard part is to find upper bounds.

Example

- The free group F_g has the obvious presentation $\langle s_1, s_2, \ldots s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely, $\operatorname{defi}(F_g) = g$.

- If G is a finite group, $\operatorname{defi}(G) \leq 0$.

- The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n = 1 \rangle$ realizes the deficiency.

- The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is -1, the obvious presentation $\langle s, t \mid s^n = t^n = [s, t] = 1 \rangle$ realizes the deficiency.

Example (Deficiency and free products)

- The deficiency is not additive under free products by the following example due to Hog-Lustig-Metzler.

- The group

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$$

  has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

- One may think that its deficiency is -2. However, it turns out that its deficiency is -1 realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

Theorem (Deficiency and group extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ be an exact sequence of infinite groups. Suppose that G is finitely presented and H is finitely generated. Then:

$$\text{defi}(G) \leq 1.$$

- An important invariant of a closed oriented $4k$-dimensional manifold M is its signature

$$\text{sign}(M) \in \mathbb{Z}$$

which is the signature of its intersection pairing.

- We have the relation $\text{sign}(M) \equiv \chi(M) \mod 2$.

- If $k = 1$, we have by the Hirzebruch signature formula

$$\text{sign}(M) = \frac{1}{3} \cdot \langle p_1(M), [M] \rangle.$$

Theorem (Signatures of 4-manifolds and group extensions, Lück)

Let M be a closed oriented 4-manifold. Suppose that $\pi_1(M)$ contains an infinite normal finitely generated subgroup of infinite index.

Then

$$|\text{sign}(M)| \leq \chi(M).$$

- Let R be a ring and let G be a group. The group ring RG, sometimes also denoted by $R[G]$, is the R-algebra, whose underlying R-module is the free R-module generated by G and whose multiplication comes from the group structure.

- An element $x \in RG$ is a formal sum $\sum_{g \in G} r_g \cdot g$ such that only finitely many of the coefficients $r_g \in R$ are different from zero.

- The multiplication comes from the tautological formula $g \cdot h = g \cdot h$, more precisely

$$\left(\sum_{g \in G} r_g \cdot g \right) \cdot \left(\sum_{g \in G} s_g \cdot g \right) := \sum_{g \in G} \left(\sum_{h,k \in G, hk=g} r_h s_k \right) \cdot g.$$

- Group rings arise in representation theory and topology as follows.

- An *RG*-module *P* is the same as *G*-representation with coefficients in *R*, i.e., a *R*-modul *P* together with a *G*-action by *R*-linear maps.

- Let $\overline{X} \to X$ be a *G*-covering of the *CW*-complex *X*, i.e., a principal *G*-bundle over *X* or, equivalently, a normal covering with *G* as group of deck transformations. An example for connected *X* is the universal covering $\widetilde{X} \to X$ with $G = \pi_1(X)$.

- Then the cellular $\mathbb{Z}$-chain complex $C_*(\overline{X})$, which is a priori a free $\mathbb{Z}$-chain complex, inherits from the *G*-action on $\overline{X}$ the structure of a free $\mathbb{Z}G$-chain complex, where the set of *n*-cells in *X* determines a $\mathbb{Z}G$-basis for $C_*(\overline{X})$.

- Group rings are in general very complicated. For instance, there is the conjecture that the complex group ring $\mathbb{C}G$ is Noetherian if and only if G is virtually poly-cyclic.

- Let us figure out whether there are idempotents x in RG, i.e., elements with $x^2 = x$.

- Here is the only known construction of an idempotent in $\mathbb{C}G$. Consider an element $g \in G$ which has finite order n. Then we can take
$$x = \frac{1}{n} \cdot \sum_{i=0}^{n-1} g^i.$$

- In particular we know no idempotent in $\mathbb{C}G$ besides 0 and 1 if G is torsionfree.

Conjecture (Idempotent Conjecture, (Kaplansky))

Let G be a torsionfree group. Then all idempotents of $\mathbb{C}G$ are trivial, i.e., equal to 0 or 1.

Conjecture (Zero-divisor Conjecture, (Kaplansky))

Let G be a torsionfree group. Then $\mathbb{C}G$ has no zero-divisors.

Conjecture (Embedding Conjecture)

Let G be a torsionfree group. Then $\mathbb{C}G$ embeds into a skew-field.

- Embedding Conjecture $\implies$ Zero-divisor Conjecture $\implies$ Idempotent Conjecture.

Definition (Projective class group $K_0(R)$)

The projective class group of a ring R

$$K_0(R)$$

is defined to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

Definition ($G_0(RG)$)

The abelian group of a ring R

$$G_0(R)$$

is defined as above but one replaces "finitely generated projective" by "finitely generated" everywhere.

Lemma

The element $[\mathbb{C}G]$ in $K_0(\mathbb{C}G)$ generates an infinite cyclic group.

Proof.

The augmentation homomorphism $\epsilon\colon \mathbb{C}G \to \mathbb{C}$ and the dimension of a complex vector space induce a homomorphism

$$K_0(\mathbb{C}G) \to \mathbb{Z}, \quad [P] \mapsto \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} P)$$

which sends $[\mathbb{C}G]$ to a generator of $\mathbb{Z}$. □

Lemma

Suppose that G contains $F_2 := \mathbb{Z} * \mathbb{Z}$ as subgroup. Then $[\mathbb{C}G] = 0$ in $G_0(\mathbb{C}G)$.

Proof.

- Let $i: F_2 \to G$ be the inclusion and $p: F_2 \to \mathbb{Z}$ be any surjective group homomorphism. Then induction with i and restriction with p induces group homomorphism

$$
\begin{aligned}
i_*: G_0(\mathbb{C}[F_2]) &\to G_0(\mathbb{C}G); \\
p^*: G_0(\mathbb{C}[\mathbb{Z}]) &\to G_0(\mathbb{C}[F_2]).
\end{aligned}
$$

- Considering the cellular $\mathbb{C}[\mathbb{Z}]$-chain complex of the universal covering of S^1 yields the exact sequence of $\mathbb{C}[\mathbb{Z}]$-modules $0 \to \mathbb{C}[\mathbb{Z}] \xrightarrow{s-1} \mathbb{C}[\mathbb{Z}] \to \mathbb{C} \to 0$ for a generator $s \in \mathbb{Z}$. Hence we get in $G_0(\mathbb{C}[\mathbb{Z}])$

$$
[\mathbb{C}] = [\mathbb{C}[\mathbb{Z}]] - [\mathbb{C}[\mathbb{Z}]] = 0.
$$

Continued.

- Considering the cellular $\mathbb{C}[F_2]$-chain complex of the universal covering of $S^1 \vee S^1$ yields the exact sequence of $\mathbb{C}[F_2]$-modules $0 \to \mathbb{C}[F_2] \oplus \mathbb{C}[F_2] \xrightarrow{(s_1-1,s_2-1)} \mathbb{C}[F_2] \to \mathbb{C} \to 0$ for the standard generators $s_1, s_2 \in F_2$. Hence we get in $G_0(\mathbb{C}[F_2])$

$$[\mathbb{C}] = -[\mathbb{C}[F_2]].$$

- We compute in $G_0(\mathbb{C}[F_2])$

$$[\mathbb{C}[G]] = i_*([\mathbb{C}[F_2]]) = -i_*([\mathbb{C}]) = -i_* \circ p^*([\mathbb{C}]) = -i_* \circ p^*(0) = 0.$$

$\square$

Theorem (On $G_0(\mathbb{C}G)$ for amenable groups G, Lück)

Suppose that G is amenable. Then the element $[\mathbb{C}G]$ in $G_0(\mathbb{C}G)$ generates an infinite cyclic group.

Conjecture (Characterization of amenability by G-theory)

- *The group G is amenable if and only if the element $[\mathbb{C}G]$ in $G_0(\mathbb{C}G)$ is non-trivial.*

- *The group G is amenable if and only if $G_0(\mathbb{C}G)$ is non-trivial.*

Conjecture (Euler characteristic and sectional curvature, Hopf)

Let M be a closed Riemannian manifold of even dimension $2n$. Then:

- *If its sectional curvature satisfied $\sec(M) \leq 0$, then $(-1)^n \cdot \chi(M) \geq 0$;*

- *If its sectional curvature satisfied $\sec(M) < 0$, then $(-1)^n \cdot \chi(M) > 0$.*

Theorem (S^1-actions and hyperbolic manifolds)

Any S^1-action on a hyperbolic closed manifold is trivial.

Theorem (Slice knots and Casson-Gordon invariants, Cochran-Orr-Teichner)

There are obstructions for knots to be slice which go far beyond the classical Casson-Gordon invariants.

Theorem (Kähler manifolds and projective algebraic varieties, Gromov)

Let M be a closed Kähler manifold, i.e., a complex manifold which comes with a so called Kähler Hermitian metric and Kähler 2-form. Suppose that it admits some Riemannian metric with negative sectional curvature, or, more generally, that $\pi_1(M)$ is hyperbolic (in the sense of Gromov) and $\pi_2(M)$ is trivial.

Then M is a projective algebraic variety.

- So far no L^2-invariants have occurred in the talk and the audience may wonder why the title contains the word L^2-invariants at all.

- The point is that the proofs of the results above or of the conjectures in certain special cases do rely on L^2-methods. The use of L^2-methods made a lot of progress possible although on the first glance they seem to be unrelated to the results and conjectures mentioned above.

- Next we give an introduction to the L^2-setting.

Basic motivation for the passage to the L^2-setting

- Given an invariant for finite *CW*-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- Examples:

Classical notion	generalized version
Homology with coefficients in $\mathbb{Z}$	Homology with coefficients in representations
Euler characteristic $\in \mathbb{Z}$	Wall's finiteness obstruction in $K_0(\mathbb{Z}\pi)$
Lefschetz numbers $\in \mathbb{Z}$	Generalized Lefschetz invariants in $\mathbb{Z}\pi_\phi$
Signature $\in \mathbb{Z}$	Surgery invariants in $L_*(\mathbb{Z}G)$
—	torsion invariants

- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X; \mathbb{C})).$$

- Here are two naive attempts which fail:
  - $\dim_{\mathbb{C}}(H_n(\widetilde{X}; \mathbb{C}))$;
  - $\dim_{\mathbb{C}\pi}(H_n(\widetilde{X}; \mathbb{C}))$,
    where $\dim_{\mathbb{C}\pi}(M)$ for a $\mathbb{C}[\pi]$-module could be chosen for instance as $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} M)$.

- The problem is that $\mathbb{C}\pi$ is in general not Noetherian and $\dim_{\mathbb{C}\pi}(M)$ is in general not additive under exact sequences.

- We will use the following successful approach which is essentially due to Atiyah.

Group von Neumann algebras

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition (Group von Neumann algebra and its trace)

- Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$.

- The von Neumann trace is defined by

$$\text{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Example (Finite G)

- Let G be a finite group.

- Then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$.

- The trace $\mathrm{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient λ_e.

- So the new approach is only interesting when G is infinite.

Example ($G = \mathbb{Z}^n$)

- Let G be $\mathbb{Z}^n$. This is an easy but very illuminating example.

- Let $L^2(T^n)$ be the Hilbert space of L^2-integrable functions $T^n \to \mathbb{C}$. The Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism

$$L^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n).$$

- Let $L^\infty(T^n)$ be the Banach space of essentially bounded measurable functions $f \colon T^n \to \mathbb{C}$. We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f$$

where $M_f \colon L^2(T^n) \to L^2(T^n)$ is the bounded $\mathbb{Z}^n$-operator $g \mapsto g \cdot f$.

- Under this identification the trace becomes

$$\mathrm{tr}_{\mathcal{N}(\mathbb{Z}^n)} \colon L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.$$

von Neumann dimension

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^n$ for some $n \geq 0$.

A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f \colon V \to W$ is a bounded G-equivariant operator.

Definition (Murray-von Neumann dimension)

Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p \colon L^2(G)^n \to L^2(G)^n$ such that there is a linear isometric G-equivariant isomorphism $\operatorname{im}(p) \xrightarrow{\cong} V$. Define the Murray-von Neumann dimension of V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in \mathbb{R}^{\geq 0}.$$

Example (Finite *G*)

- Let *G* be a finite group.

- A finitely generated Hilbert $\mathcal{N}(G)$-module *V* is the same as a unitary finite dimensional *G*-representation.

- We get for its von Neumann dimension

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)

- Let G be $\mathbb{Z}^n$.

- Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} \colon L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X.

- Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \mathrm{vol}(X).$$

- In particular each $r \in \mathbb{R}^{\geq 0}$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$.

Definition (Weakly exact)

- A sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is weakly exact at V if the kernel $\ker(p)$ of p and the closure $\overline{\mathrm{im}(i)}$ of the image $\mathrm{im}(i)$ of i agree.

- A map of Hilbert $\mathcal{N}(G)$-modules $f \colon V \to W$ is a weak isomorphism if it is injective and has dense image.

Example (Finite G)

If G is finite, weakly exact is the same as exact and weak isomorphism is the same as isomorphism.

Example

- The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

  $$M_{z-1} \colon L^2(\mathbb{Z}) = L^2(S^1) \to L^2(\mathbb{Z}) = L^2(S^1), \quad u(z) \mapsto (z-1) \cdot u(z)$$

  is a weak isomorphism, but not an isomorphism.

- Its kernel is trivial. Its image is dense but not equal to $L^2(S^1)$.

Theorem (Properties of the Murray-von Neumann dimension)

1 Faithfulness

We have for a finitely generated Hilbert $\mathcal{N}(G)$-module V

$$V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0;$$

2 Additivity

If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then

$$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V);$$

3 Cofinality

Let $\{ V_i \mid i \in I \}$ be a directed system of Hilbert $\mathcal{N}(G)$- submodules of V, directed by inclusion. Then

$$\dim_{\mathcal{N}(G)}\left(\overline{\bigcup_{i \in I} V_i} \right) = \sup\{\dim_{\mathcal{N}(G)}(V_i) \mid i \in I\}.$$

L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

- Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

- Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

- Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

- Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

Theorem (Main properties of L^2-Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{Y});$$

- *Euler-Poincaré formula*

  We have

  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X});$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Example (Finite π)

If π is finite then

$$b_n^{(2)}(\widetilde{X}) = \frac{b_n(\widetilde{X})}{|\pi|}.$$

Example (S^1)

Consider the $\mathbb{Z}$-CW-complex $\widetilde{S^1}$. We get for $C_*^{(2)}(\widetilde{S^1})$

$$\ldots \to 0 \to L^2(\mathbb{Z}) \xrightarrow{M_{z-1}} L^2(\mathbb{Z}) \to 0 \to \ldots$$

and hence $H_n^{(2)}(\widetilde{S^1}) = 0$ and $b_n^{(2)}(\widetilde{S^1}) = 0$ for all ≥ 0.

Example (L^2-Betti number of surfaces)

- Let F_g be the orientable closed surface of genus $g \geq 1$.

- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.

- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.

- Since $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.

- The Euler-Poincaré formula shows

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g - 2; \\ b_n^{(2)}(\widetilde{F_g}) &= 0 \quad \text{for } n \neq 1. \end{aligned}$$

Example ($\pi = \mathbb{Z}^d$)

- Let X be a connected *CW*-complex of finite type with fundamental group $\mathbb{Z}^d$.

- Let $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ be the quotient field of the commutative integral domain $\mathbb{C}[\mathbb{Z}^d]$.

- Then
$$b_n^{(2)}(\widetilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\widetilde{X}) \right).$$

- Obviously this implies
$$b_n^{(2)}(\widetilde{X}) \in \mathbb{Z}.$$

- For a discrete group *G* we can consider more generally any free finite *G-CW*-complex $\overline{X}$ which is the same as a *G*-covering $\overline{X} \to X$ over a finite *CW*-complex *X*. (Actually proper finite *G-CW*-complex suffices.)

- The universal covering $p \colon \widetilde{X} \to X$ over a connected finite *CW*-complex is a special case for $G = \pi_1(X)$.

- Then one can apply the same construction to the finite free $\mathbb{Z}G$-chain complex $C_*(\overline{X})$. Thus we obtain the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\overline{X}; \mathcal{N}(G)) := H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})),$$

and define

$$b_n^{(2)}(\overline{X}; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\left(H_n^{(2)}(\overline{X}; \mathcal{N}(G))\right) \in \mathbb{R}^{\geq 0}.$$

- Let $i\colon H \to G$ be an injective group homomorphism and C_* be a finite free $\mathbb{Z}H$-chain complex.

- Then $i_* C_* := \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*$ is a finite free $\mathbb{Z}G$-chain complex.

- We have the following formula

$$\dim_{\mathcal{N}(G)}\bigl(H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} i_* C_*)\bigr) = \dim_{\mathcal{N}(H)}\bigl(H_n^{(2)}(L^2(H) \otimes_{\mathbb{Z}H} C_*)\bigr).$$

Lemma

If $\overline{X}$ is a finite free H-CW-complex, then we get for $i_* X := G \times_H \overline{X}$

$$b_n^{(2)}(i_* \overline{X}; \mathcal{N}(G)) = b_n^{(2)}(\overline{X}; \mathcal{N}(H)).$$

- The corresponding statement is wrong if we drop the condition that i is injective.

- An example comes from $p\colon \mathbb{Z} \to \{1\}$ and $\widetilde{X} = \widetilde{S^1}$ since then $p_*\widetilde{S^1} = S^1$ and we have for $n = 0, 1$

$$b_n^{(2)}(\widetilde{S^1}; \mathcal{N}(\mathbb{Z})) = b_n^{(2)}(\widetilde{S^1}) = 0,$$

and

$$b_n^{(2)}(p_*\widetilde{S^1}; \mathcal{N}(\{1\})) = b_n(S^1) = 1.$$

The L^2-Mayer Vietoris sequence

Lemma

Let $0 \to C_*^{(2)} \xrightarrow{i_*^{(2)}} D_*^{(2)} \xrightarrow{p_*^{(2)}} E_*^{(2)} \to 0$ *be a weakly exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes.*

Then there is a long weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\cdots \xrightarrow{\delta_{n+1}^{(2)}} H_n^{(2)}(C_*^{(2)}) \xrightarrow{H_n^{(2)}(i_*^{(2)})} H_n^{(2)}(D_*^{(2)}) \xrightarrow{H_n^{(2)}(p_*^{(2)})} H_n^{(2)}(E_*^{(2)})$$

$$\xrightarrow{\delta_n^{(2)}} H_{n-1}^{(2)}(C_*^{(2)}) \xrightarrow{H_{n-1}^{(2)}(i_*^{(2)})} H_{n-1}^{(2)}(D_*^{(2)})$$

$$\xrightarrow{H_{n-1}^{(2)}(p_*^{(2)})} H_{n-1}^{(2)}(E_*^{(2)}) \xrightarrow{\delta_{n-1}^{(2)}} \cdots .$$

Lemma

Let

$$\begin{array}{ccc} \overline{X}_0 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ \overline{X}_2 & \longrightarrow & \overline{X} \end{array}$$

be a cellular G-pushout of finite free G-CW-complexes, i.e., a G-pushout, where the upper arrow is an inclusion of a pair of free finite G-CW-complexes and the left vertical arrow is cellular.

Then we obtain a long weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\cdots \to H_n^{(2)}(\overline{X_0}; \mathcal{N}(G)) \to H_n^{(2)}(\overline{X_1}; \mathcal{N}(G)) \oplus H_n^{(2)}(\overline{X_2}; \mathcal{N}(G))$$
$$\to H_n^{(2)}(\overline{X}; \mathcal{N}(G)) \to H_{n-1}^{(2)}(\overline{X_0}; \mathcal{N}(G))$$
$$\to H_{n-1}^{(2)}(\overline{X_1}; \mathcal{N}(G)) \oplus H_{n-1}^{(2)}(\overline{X_2}; \mathcal{N}(G)) \to H_{n-1}^{(2)}(\overline{X}; \mathcal{N}(G)) \to \cdots .$$

Proof.

- From the cellular G-pushout we obtain an exact sequence of $\mathbb{Z}G$-chain complexes

$$0 \to C_*(\overline{X}_0) \to C_*(\overline{X}_1) \oplus C_*(\overline{X}_2) \to C_*(\overline{X}) \to 0.$$

- It induces an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes

$$0 \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_0) \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_1) \oplus L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_2)$$
$$\to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}) \to 0.$$

- Now apply the previous result.

$\square$

Definition (L^2-acyclic)

A finite (not necessarily connected) *CW*-complex X is called L^2-acyclic, if $b_n^{(2)}(\widetilde{C}) = 0$ holds for every $C \in \pi_0(X)$ and $n \in \mathbb{Z}$.

- If X is a finite (not necessarily connected) *CW*-complex, we define

$$b_n^{(2)}(\widetilde{X}) := \sum_{C \in \pi_0(X)} b_n^{(2)}(\widetilde{C}) \in \mathbb{R}^{\geq 0}.$$

Definition (π_1-injective)

A map $X \to Y$ is called π_1-injective, if for every choice of base point in X the induced map on the fundamental groups is injective.

- Consider a cellular pushout of finite *CW*-complexes

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X
\end{array}
$$

such that each of the maps $X_i \to X$ is π_1-injective.

Lemma

We get under the assumptions above for any $n \in \mathbb{Z}$

- *If X_0 is L^2-acyclic, then*

$$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{X_1}) + b_n^{(2)}(\widetilde{X_2}).$$

- *If X_0, X_1 and X_2 are L^2-acyclic, then X is L^2-acyclic.*

Proof.

- Without loss of generality we can assume that X is connected.

- By pulling back the universal covering $\widetilde{X} \to X$ to X_i, we obtain a cellular $\pi = \pi_1(X)$-pushout

$$\begin{array}{ccc} \overline{X}_0 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ \overline{X}_2 & \longrightarrow & \widetilde{X} \end{array}$$

- Notice that $\overline{X}_i$ is in general not the universal covering of X_i.

□

Proof continued.

- Because of the associated long exact L^2-sequence and the weak exactness of the von Neumann dimension, it suffices to show for $n \in \mathbb{Z}$ and $i = 1, 2$

$$
\begin{aligned}
H_n^{(2)}(\overline{X_0}; \mathcal{N}(\pi)) &= 0; \\
b_n^{(2)}(\overline{X_i}; \mathcal{N}(\pi)) &= b_n^{(2)}(\widetilde{X}_i).
\end{aligned}
$$

- This follows from π_1-injectivity, the lemma above about L^2-Betti numbers and induction, the assumption that X_0 is L^2-acyclic, and the faithfulness of the von Neumann dimension.

$\square$

Some computations and results

Example (Finite self-coverings)

- We get for a connected CW-complex X of finite type, for which there is a self-covering $X \to X$ with d-sheets for some integer $d \geq 2$,
$$b_n^{(2)}(\widetilde{X}) = 0 \quad \text{for } n \geq 0.$$

- This is certainly not true for the classical Betti numbers $b_n(X)$.

- This implies for each connected CW-complex Y of finite type that $S^1 \times Y$ is L^2-acyclic.

- The latter equation follows also from the Künneth formula for L^2-Betti numbers and the previous computation that $b_n^{(2)}(\widetilde{S^1})$ vanishes for all n.

Theorem (S^1-actions, Lück)

Let M be a connected compact manifold with S^1-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to M$, $z \mapsto zx$ is π_1-injective.

Then M is L^2-acyclic.

Proof.

Each of the S^1-orbits S^1/H in M satisfies $S^1/H \cong S^1$. Now use induction over the number of cells $S^1/H_i \times D^n$ and a previous result using π_1-injectivity and the vanishing of the L^2-Betti numbers of spaces of the shape $S^1 \times X$. □

Theorem (S^1-actions on aspherical manifolds, Lück)

Let M be an aspherical closed manifold with non-trivial S^1-action. Then

1. *The action has no fixed points;*
2. *The map $S^1 \to M$, $z \mapsto zx$ is π_1-injective for $x \in M$;*
3. *$b_n^{(2)}(\widetilde{M}) = 0$ for $n \geq 0$ and $\chi(M) = 0$.*

Proof.

The hard part is to show that the second assertion holds, since M is aspherical. Then the first assertion is obvious and the third assertion follows from the previous theorem. □

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0, \ ||\widetilde{\omega}||_{L^2} < \infty\}.$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels)

$$b^{(2)}_n(\widetilde{M}) = \lim_{t \to \infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, d\mathrm{vol}.$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

Theorem (Hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & \text{, if } 2n \neq d; \\ > 0 & \text{, if } 2n = d. \end{cases}$$

Proof.

- Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$.

- A direct computation shows that $\mathcal{H}_{(2)}^n(\mathbb{H}^d)$ is not zero if and only if $2n = d$.

- Hence $b_n^{(2)}(\widetilde{M}) = \dim_{\mathcal{N}(\pi)}(\mathcal{H}_{(2)}^p(\widetilde{M}))$ is not zero if and only if $2n = d$.

$\square$

Corollary

Let M be a hyperbolic closed manifold of dimension d. Then

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Proof.

(1) We get from the Euler-Poincaré formula and the last result

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

(2) We give the proof only for $d = 2m$ even. Then $b_m^{(2)}(\widetilde{M}) > 0$. Since $\widetilde{M} = \mathbb{H}^d$ is contractible, M is aspherical. Now apply a previous result about S^1-actions. □

Theorem (3-manifolds, Lott-Lück)

Let the 3-manifold M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then

$$
\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M) \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2.
\end{aligned}
$$

Proof.

- We have already explained why a closed hyperbolic 3-manifold is L^2-acyclic.

- One of the hard parts of the proof is to show that this is also true for any hyperbolic 3-manifold with incompressible toral boundary.

- Notice that these have finite volume.

- One has to introduce appropriate boundary conditions and Sobolev theory to write down the relevant analytic L^2-deRham complexes and L^2-Laplace operators.

- A key ingredient is the decomposition of such a manifold into its core and a finite number of cusps.

$\square$

Proof continued.

- This can be used to write the L^2-Betti number as an integral over a fundamental domain $\mathcal{F}$ of finite volume, where the integrand is given by data depending on $\mathbb{IH}^3$ only:

$$b_n^{(2)}(\widetilde{M}) = \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, d\text{vol} \, .$$

- Since $\mathbb{H}^3$ has a lot of symmetries, the integrand does not depend on $\tilde{x}$ and is a constant C_n depending only on $\mathbb{IH}^3$.

- Hence we get

$$b_n^{(2)}(\widetilde{M}) = C_n \cdot \text{vol}(M).$$

- From the closed case we deduce $C_n = 0$.

$\square$

Proof continued.

- Next we show that any Seifert manifold with infinite fundamental group is L^2-acyclic.

- This follows from the fact that such a manifold is finitely covered by the total space of an S^1-bundle $S^1 \to E \to F$ over a surface with injective $\pi_1(S^1) \to \pi_1(E)$ using previous results.

- In the next step one shows that any irreducible 3-manifold M with incompressible or empty boundary and infinite fundamental group is L^2-acyclic.

- By the Thurston Geometrization Conjecture we can find a family of incompressible tori which decompose M into hyperbolic and Seifert pieces. The tori and all these pieces are L^2-acyclic.

- Now the claim follows from the L^2-Mayer Vietoris sequence.

Proof continued.

- In the next step one shows that any irreducible 3-manifold M with incompressible boundary and infinite fundamental group satisfies $b_1^{(2)}(\widetilde{M}) = -\chi(M)$ and $b_n^{(2)}(\widetilde{M}) = 0$ for $n \neq 1$.

- This follows by considering $N = M \cup_{\partial M} M$ using the L^2-Mayer-Vietoris sequence, the already proved fact that N is L^2-acyclic and the previous computation of the L^2-Betti numbers for surfaces.

- In the next step one shows that any irreducible 3-manifold M with infinite fundamental group satisfies $b_1^{(2)}(\widetilde{M}) = -\chi(M)$ and $b_n^{(2)}(\widetilde{M}) = 0$ for $n \neq 1$.

$\square$

Proof continued.

- This is reduced by an iterated application of the Loop Theorem to the case where the boundary is incompressible. Namely, using the Loop Theorem one gets an embedded disk $D^2 \subseteq M$ along which one can decompose M as $M_1 \cup_{D^2} M_2$ or as $M_1 \cup_{S^0 \times D^2} D^1 \times D^2$ depending on whether D^2 is separating or not.

- Since the only prime 3-manifold that is not irreducible is $S^1 \times S^2$, and every manifold M with finite fundamental group satisfies the result by a direct inspection of the Betti numbers of its universal covering, the claim is proved for all prime 3-manifolds.

- Finally one uses the L^2-Mayer Vietoris sequence to prove the claim in general using the prime decomposition.

$\square$

Corollary

Let M be a 3-manifold. Then M is L^2-acyclic if and only if one of the following cases occur:

- *M is an irreducible 3-manifold with infinite fundamental group whose boundary is empty or toral.*
- *M is $S^1 \times S^2$ or $\mathbb{RP}^3 \sharp \mathbb{RP}^3$.*

Corollary

Let M be a compact n-manifold such that $n \leq 3$ and its fundamental group is torsionfree.

Then all its L^2-Betti numbers are integers.

Theorem (Mapping tori, Lück)

Let $f\colon X \to X$ be a cellular self-homotopy equivalence of a connected CW-complex X of finite type. Let T_f be the mapping torus. Then

$$b_n^{(2)}(\widetilde{T}_f) = 0 \quad \text{for } n \geq 0.$$

Proof.

- As $T_{f^d} \to T_f$ is up to homotopy a d-sheeted covering, we get

$$b_n^{(2)}(\widetilde{T}_f) = \frac{b_n^{(2)}(\widetilde{T_{f^d}})}{d}.$$

$\square$

Proof continued.

- If $\beta_n(X)$ is the number of n-cells, then there is up to homotopy equivalence a CW-structure on T_{f^d} with $\beta_n(T_{f^d}) = \beta_n(X) + \beta_{n-1}(X)$. We have

$$b_n^{(2)}(\widetilde{T_{f^d}}) = \dim_{\mathcal{N}(G)} \left(H_n^{(2)}(C_n^{(2)}(\widetilde{T_{f^d}})) \right)$$
$$\leq \dim_{\mathcal{N}(G)} \left(C_n^{(2)}(\widetilde{T_{f^d}}) \right) = \beta_n(T_{f^d}).$$

- This implies for all $d \geq 1$

$$b_n^{(2)}(\widetilde{T_f}) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$

- Taking the limit for $d \to \infty$ yields the claim.

□

- Let M be an irreducible manifold M with infinite fundamental group and empty or incompressible toral boundary which is not a closed graph manifold.

- Agol proved the Virtually Fibering Conjecture for such M saying that a finite covering of M is a mapping torus.

- This implies by the result above that M is L^2-acyclic.

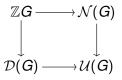# The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$
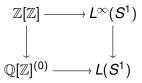
- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, its is given by

$$
\begin{array}{ccc}
\mathbb{Z}G & \longrightarrow & \mathbb{C}G \\
\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\
\mathbb{Q}G & \longrightarrow & \mathbb{C}G
\end{array}
$$

- If $G = \mathbb{Z}$, it is given by

$$
\begin{array}{ccc}
\mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\
\downarrow & & \downarrow \\
\mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow & L(S^1)
\end{array}
$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

  $$\dim_{\mathcal{N}(G)}\big(\ker(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n)\big)$$

  is an integer. In this case this dimension agrees with

  $$\dim_{\mathcal{D}(G)}\big(\ker(r_A \colon \mathcal{D}(G)^m \to \mathcal{D}(G)^n)\big).$$

- The general version above is equivalent to the one stated before if G is finitely presented.

- Obviously the Atiyah Conjecture implies the Embedding Conjecture and hence the Zero-divisor Conjecture and the Idempotent Conjecture due to Kaplansky.

- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

- However, there exist closed Riemannian manifolds whose universal coverings have an L^2-Betti number which is irrational, see Austin, Grabowski.

Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under extensions with elementary amenable groups as quotients and under directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*

2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- A group is called locally indicable if every non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$. Examples are one-relator-groups.

Theorem (Jaikin-Zapirain & Lopez-Alvarez)

If G is locally indicable, then it satisfies the Atiyah Conjecture.

Theorem (Strategy to prove the Atiyah Conjecture I, Linnell)

Let G be a torsionfree group. Then the Atiyah Conjecture holds if the following three assertions are true:

- *The map $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective;*

- *The map $K_0(\mathbb{C}G) \to K_0(\mathcal{D}(G))$ is surjective;*

- *The ring $\mathcal{D}(G)$ is semisimple.*

Theorem (Strategy to prove the Atiyah Conjecture II)

Let G be a torsionfree group. Then the Atiyah Conjecture holds if the following two assertions are true, where $\mathcal{R}(G) \subseteq \mathcal{U}(G)$ is the smallest $*$-regular subring of the $*$-regular ring $\mathcal{U}(G)$.

- *The map $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective;*

- *The map $K_0(\mathbb{C}G) \to K_0(\mathcal{R}(G))$ is surjective.*

Theorem

Let G be a torsionfree group. Then the following assertions are equivalent:

- *The Atiyah Conjecture holds for G;*

- $\mathcal{D}(G)$ *is a skew-field;*

- $\mathcal{R}(G)$ *is a skew-field.*

- If the torsionfree group *G* satisfies the Atiyah Conjecture, then $\mathcal{D}(G) = \mathcal{R}(G)$.

- Recall that for a torsionfree group the Atiyah Conjecture predicts that for a closed Riemannian manifold with G as fundamental group the following integral is an integer

$$\lim_{t \to \infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, d\mathrm{vol}_{\widetilde{M}}.$$

- By the last Theorems it is equivalent to the K-theoretic statement that $\widetilde{K}_0(\mathcal{R}(G))$ vanishes and to the ring theoretic statement that $\mathcal{D}(G)$ is a skew-field.

- The proof that these three facts imply the Atiyah Conjecture is rather involved. It is based on the fundamental square and the fact that the generalized dimension function for $\mathcal{N}(G)$ which we will introduce later, extends to an appropriate dimension function $\dim_{\mathcal{U}(G)}$ for $\mathcal{U}(G)$-modules such that for any $\mathcal{N}(G)$-module M we have

$$\dim_{\mathcal{N}(G)}(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M).$$

- In general there are no relations between the Betti numbers $b_n(X)$ and the L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

- Given an integer $l \geq 1$ and a sequence $r_1, r_2, \ldots, r_l$ of non-negative rational numbers, we can construct a group G such that BG is of finite type and

$$
\begin{array}{rcll}
b_n^{(2)}(BG) & = & r_n & \text{for } 1 \leq n \leq l; \\
b_n^{(2)}(BG) & = & 0 & \text{for } l+1 \leq n; \\
b_n(BG) & = & 0 & \text{for } n \geq 1.
\end{array}
$$

- For any sequence $s_1, s_2, \ldots$ of non-negative integers there is a *CW*-complex X of finite type such that for $n \geq 1$

$$
\begin{array}{rcl}
b_n(X) & = & s_n; \\
b_n^{(2)}(\widetilde{X}) & = & 0.
\end{array}
$$

Theorem (Approximation Theorem, Lück)

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

- Ordinary Betti numbers are not multiplicative under finite coverings, whereas the L^2-Betti numbers are. With the expression

$$\lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]},$$

we try to force the Betti numbers to be multiplicative by a limit process.

- The theorem above says that L^2-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

- There are generalizations of the Approximation Theorem above where one drops the condition that $[G : G_i]$ is finite. They allow to extend the class of groups for which the Atiyah Conjecture holds, considerably. The trick is that a limit of a convergent sequence of integers is again an integer.

Applications to deficiency and signature

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\text{defi}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

Lemma

Let G be a finitely presented group. Then

$$\text{defi}(G) \leq 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation P that

$$g(P) - r(P) \leq 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a *CW*-complex realizing P. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{aligned} b_n^{(2)}(\widetilde{X}) &= b_n^{(2)}(G) \quad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) &\geq b_2^{(2)}(G). \end{aligned}$$

Theorem (Deficiency and extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups.*
Suppose that G is finitely presented and H is finitely generated. Then:

1. $b_1^{(2)}(G) = 0$;
2. $\mathrm{defi}(G) \leq 1$;
3. *Let M be a closed oriented 4-manifold with G as fundamental group. Then*

$$|\operatorname{sign}(M)| \leq \chi(M).$$

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.

- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.

- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies in the case $\dim(M) = 2n$

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M})$$

and hence the Hopf Conjecture.

- The Singer Conjecture gives also evidence for the Atiyah Conjecture.

Definition (Kähler hyperbolic manifold)

A Kähler hyperbolic manifold is a closed connected Kähler manifold M whose fundamental form ω is $\widetilde{d}$(bounded), i.e. its lift $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ to the universal covering can be written as $d(\eta)$ for some bounded 1-form $\eta \in \Omega^1(\widetilde{M})$.

Theorem (Gromov)

Let M be a closed Kähler hyperbolic manifold of complex dimension c. Then

$$
\begin{aligned}
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{if } n \neq c; \\
b_c^{(2)}(\widetilde{M}) &> 0; \\
(-1)^c \cdot \chi(M) &> 0;
\end{aligned}
$$

- Let M be a closed Kähler manifold. It is Kähler hyperbolic if it admits some Riemannian metric with negative sectional curvature, or, if, more generally, $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial.

- This gives some evidence for the Singer Conjecture.

- Next we sketch the proof of Gromov that M is an algebraic variety.

  Namely, we conclude that $\mathcal{H}^c_{(2)}(\widetilde{M})$ is non-trivial. Using Hodge theory one can construct a non-trivial holomorphic L^2-integrable c-form on $\widetilde{M}$. This implies the existence of appropriate meromorphic functions on M. This leads to the result that the canonical line bundle L is quasi-ample and M is Moishezon. Hence M can be holomorphically embedded into $\mathbb{CP}^N$ for large N. This implies that M is an algebraic variety.

The generalized dimension function

- Next we present a purely algebraic approach to L^2-Betti numbers.

- Recall that by definition

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \mathrm{mor}_{\mathcal{N}(G)}(L^2(G), L^2(G)).$$

- This induces a bijection of $\mathbb{C}$-vector spaces

$$M(m, n, \mathcal{N}(G)) \xrightarrow{\cong} \mathrm{mor}_{\mathcal{N}(G)}(L^2(G)^m, L^2(G)^n).$$

- It is compatible with multiplication of matrices and composition of morphisms. This extends to finitely generated Hilbert $\mathcal{N}(G)$-modules and finitely projective $\mathcal{N}(G)$-modules by means of idempotent completion as follows.

Theorem (Modules over $\mathcal{N}(G)$ and Hilbert $\mathcal{N}(G)$-modules)

We obtain an equivalence of $\mathbb{C}$-categories

$$\nu \colon \{\text{fin. gen. proj. } \mathcal{N}(G)\text{-mod.}\} \xrightarrow{\simeq} \{\text{fin. gen. Hilb. } \mathcal{N}(G)\text{-mod.}\}.$$

Definition (Closure and weak exactness for modules)

- Let R be a ring. Let M be an R-submodule of N.

- Define the closure of M in N to be the R-submodule of N

$$\overline{M} = \{x \in N \mid f(x) = 0 \text{ for all } f \in N^* \text{ with } M \subset \ker(f)\}.$$

- For an R-module M define the R-submodule $\mathbf{T}M$ and the R-quotient module $\mathbf{P}M$ by:

$$\mathbf{T}M := \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\};$$
$$\mathbf{P}M := M/\mathbf{T}M.$$

- We call a sequence of R-modules $L \xrightarrow{i} M \xrightarrow{q} N$ weakly exact if $\overline{\operatorname{im}(i)} = \ker(q)$.

- Notice that **T**M is the closure of the trivial submodule in M.

- It can also be described as the kernel of the canonical map

$$i(M)\colon M \to (M^*)^*$$

  which sends $x \in M$ to the map $M^* \to R \quad f \mapsto f(x)^*$.

- Notice that **TP**$M = 0$ and that **P**$M = 0$ is equivalent to $M^* = 0$.

Example ($R = \mathbb{Z}$)

- Let $R = \mathbb{Z}$. Let M be a finitely generated $\mathbb{Z}$-module and $K \subset M$.

- Then

$$
\begin{aligned}
\overline{K} &= \{x \in M \mid n \cdot x \in K \text{ for some } n \in \mathbb{Z}\}; \\
\mathbf{T}M &:= \operatorname{tors}(M); \\
\mathbf{P}M &= M/\operatorname{tors}(M).
\end{aligned}
$$

- A sequence $M_0 \to M_1 \to M_2$ of finitely generated $\mathbb{Z}$-modules is weakly exact if and only if it is exact after applying $\mathbb{Q} \otimes_{\mathbb{Z}} -$.

- Notice that $\mathbf{P}M$ is finitely generated free and we have

$$
\begin{aligned}
\operatorname{rk}_{\mathbb{Z}}(\mathbf{T}M) &= 0; \\
\operatorname{rk}_{\mathbb{Z}}(M) &= \operatorname{rk}_{\mathbb{Z}}(\mathbf{P}M).
\end{aligned}
$$

Definition (Hattori-Stallings rank)

Let P be a finitely generated projective $\mathcal{N}(G)$-module. Choose a matrix $A \in M_n(\mathcal{N}(G))$ with $A^2 = A$ such that the image of $r_A \colon \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ is $\mathcal{N}(G)$-isomorphic to P. Define

$$\dim_{\mathcal{N}(G)}(P) := \operatorname{tr}_{\mathcal{N}(G)}(A) \quad [0, \infty).$$

Lemma

1. *The functors ν and ν^{-1} preserve exact sequences and weakly exact sequences;*

2. *If P is a finitely generated projective $\mathcal{N}(G)$-module, then*

$$\dim_{\mathcal{N}(G)}(P) = \dim_{\mathcal{N}(G)}(\nu(P)).$$

- Next we want to investigate the ring theoretic properties of $\mathcal{N}(G)$ (after forgetting the topology).

- $\mathcal{N}(G)$ is Noetherian if and only if G is finite.

- $\mathcal{N}(G)$ contains zero-divisors if G is non-trivial.

Definition (Semi-hereditary)

A ring R is called semihereditary if any finitely generated submodule of a projective module is projective.

Lemma

$\mathcal{N}(G)$ is semihereditary.

- The following results and definitions can be understood by the slogan that $\mathcal{N}(G)$ behaves like $\mathbb{Z}$ if one forgets that $\mathbb{Z}$ is Noetherian and has no zero-divisors. In this sense all properties of $\mathbb{Z}$ carry over to $\mathcal{N}(G)$.

Lemma

Let M be a finitely generated $\mathcal{N}(G)$-module. Then

1. *Let $K \subset M$ be a submodule. Then $\overline{K} \subset M$ is a direct summand and $M/\overline{K}$ is finitely generated projective;*

2. $\mathbf{P}M$ *is a finitely generated projective $\mathcal{N}(G)$-module and we get a splitting*

$$M \cong \mathbf{T}M \oplus \mathbf{P}M.$$

3. *If M is finitely presented, then there is an exact sequence*

$$0 \to \mathcal{N}(G)^n \to \mathcal{N}(G)^n \to \mathbf{T}M \to 0.$$

Theorem (Dimension function for arbitrary $\mathcal{N}(G)$-modules, Lück)

There is precisely one dimension function

$$\dim \colon \{\mathcal{N}(G) - \text{modules}\} \to [0, \infty]$$

which has the following properties;

- *Extension Property*
  If M is a finitely generated projective $\mathcal{N}(G)$-module, then $\dim(M)$ agrees with the previously defined notion given by the Hattori-Stallings rank;

- *Additivity*
  If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of R-modules, then
  $$\dim(M_1) = \dim(M_0) + \dim(M_2);$$

Theorem (Continued)

- *Cofinality*

  Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e.
  $M = \cup_{i \in I} M_i$ and for two indices i and j there is an index k in I
  satisfying $M_i, M_j \subset M_k$. Then

  $$\dim(M) = \sup\{\dim(M_i) \mid i \in I\};$$

- *Continuity*

  If $K \subset M$ is a submodule of the finitely generated R-module M,
  then

  $$\dim(K) = \dim(\overline{K});$$

- *Dimension and Torsion*

  If M is a finitely generated $\mathcal{N}(G)$-module, then

  $$\dim(M) = \dim(\mathbf{P}M);$$
  $$\dim(\mathbf{T}M) = 0.$$

Proof.

- We only give the proof of uniqueness which leads to the definition of the generalized dimension function. The hard part is to prove that this definition has all the desired properties.

- Any $\mathcal{N}(G)$-module M is the colimit over the directed system of its finitely generated submodules $\{M_i \mid i \in I\}$. Hence by Cofinality

$$\dim = \sup\{\dim(M_i) \mid i \in I\}.$$

- We get for each M_i from Additivity

$$\dim(M_i) = \dim(\mathbf{P}M_i).$$

- Hence we get

$$\dim(M) = \sup\{\dim(P) \mid P \subset M \text{ finitely generated projective}\}.$$

- In the case, where the ground ring is $\mathbb{Z}$, the dimension function constructed above corresponds to assigning to a $\mathbb{Z}$-module M the element $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M) \in \{0, 1, 2, \ldots\} \amalg \{\infty\}$.

- This is not very helpful since very often the value is ∞.

- This is different for $\mathcal{N}(G)$. For instance, consider a sequence $(M_n)_{n \geq 1}$ of finitely generated projective $\mathcal{N}(G)$-modules with $\dim(M_i) = \dim_{\mathcal{N}(G)}(M_i) = 2^{-i}$. Then

$$\dim\left(\bigoplus_{i \in I} M_i\right) = \sum_{i=1}^{\infty} 2^{-i} = 1.$$

- In the sequel we write $\dim = \dim_{\mathcal{N}(G)}$.

Definition (L^2-Betti numbers for arbitrary G-spaces)

- Let X be a G-space. Its homology with coefficients in $\mathcal{N}(G)$ is

$$H_n^G(X; \mathcal{N}(G)) = H_n\left(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)\right).$$

- Define the n-th L^2-Betti number of X by

$$b_p^{(2)}(X; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\left(H_p^G(X; \mathcal{N}(G))\right) \quad \in [0, \infty].$$

Lemma

Let X be a free G-CW-complex of finite type. Then new definition above of $b_n^{(2)}(X; \mathcal{N}(G))$ agrees with the previous one.

Definition (L^2-Betti numbers of groups)

The n-th L^2-Betti number of a group G is defined to be

$$b_n^{(2)}(G) := b_n^{(2)}(EG, \mathcal{N}(G)).$$

- Next we investigate flatness properties of the von Neumann algebra $\mathcal{N}(G)$ over $\mathbb{C}[\mathbb{Z}]$.

- Since $\mathbb{C}[\mathbb{Z}]$ is a principal ideal domain, it is not hard to show that $\mathcal{N}(G)$ is flat over $\mathbb{C}[\mathbb{Z}]$.

- Since every natural number is invertible in $\mathcal{N}(G)$, one easily checks that this implies that $\mathcal{N}(G)$ is flat over $\mathbb{C}[\mathbb{Z}]$ if and only if G is virtually cyclic.

- There is a lot of evidence for the following conjecture:

Conjecture (Flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$)

$\mathcal{N}(G)$ *is flat over* $\mathbb{C}G$ *if and only if* G *is virtually cyclic.*

Theorem (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$ for amenable G, Lück)

Let G be amenable. Then $\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$, i.e., for every $\mathbb{C}G$-module M we get

$$\dim_{\mathcal{N}(G)} \left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathcal{N}(G), M) \right) = 0 \qquad \text{for } p \geq 1,$$

where we consider $\mathcal{N}(G)$ as an $\mathcal{N}(G)$-$\mathbb{C}G$-bimodule.

Lemma

If G contains $F_2 = \mathbb{Z} * \mathbb{Z}$ as subgroup, then $\mathcal{N}(G)$ is not dimension-flat over $\mathbb{C}G$.

Conjecture (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$)

$\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$ if and only if G is amenable.

Theorem (Consequences of dimension-flatness)

Suppose that $\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$.

1. *We get for any $\mathbb{C}G$-chain complex C_**

$$\dim_{\mathcal{N}(G)}\big(\mathcal{N}(G) \otimes_{\mathbb{C}G} H_n(C_*)\big) = \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{C}G} C_*)\big);$$

2. *If G is infinite and the group Γ contains G as normal subgroup, then we get for $n \geq 0$*

$$b_n^{(2)}(\Gamma) = 0.$$

Moreover, if there is a finite model for $B\Gamma$, then $\chi(B\Gamma) = 0$;

3. *We obtain a well-defined homomorphism of abelian groups*

$$G_0(\mathbb{C}G) \to \mathbb{R}, \quad [M] \mapsto \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M)$$

which sends $[\mathbb{C}G]$ to 1. This implies that the element $[\mathbb{C}G]$ in $G_0(\mathbb{C}G)$ has infinite order.

Outlook

- Unfortunately, there are a lot of very interesting aspects and very deep results by many people, which we have not covered.

- At least we want to mention some highlights.

- Gaboriau showed that the L^2-Betti numbers are (up to scaling) invariants of the measure equivalences class.

- Using L^2-Betti numbers and Gaboriau's ideas Popa solved some prominent outstanding problems about von Neumann algebras.

- Connes-Shlyakhtenko have defined L^2-Betti numbers for finite von Neumann algebras using Hochschild homology and the generalized dimensions function. If one can show that their definition applied to $\mathcal{N}(G)$ agrees with the L^2-Betti numbers of G, this would lead to a positive solution to the outstanding problem whether two finitely generated free groups are isomorphic if and only if their group von Neumann algebras are isomorphic. This is important for free probability theory.

- There is the notion of L^2-torsion due to Lück-Rothenberg in the topological and to Lott, Mathai in the analytic setting. These are the L^2-analogues of Reidemeister torsion and analytic Ray-Singer torsion. They have been identified by Burghelea-Friedlander-Kappeller-McDonald.

- There is conjecture due to Bergeron-Venkatesh which is an analogue to the Approximation Theorem for L^2-Betti numbers for the L^2-torsion in terms of torsion homology growth.

- Another open question is whether the Approximation Theorem for L^2-Betti numbers holds in prime characteristic.

- L^2-torsion and Fuglede-Kadison determinants have been linked to entropy by Deninger & Li-Thom.

- Universal L^2-torsion has been defined by Friedl-Lück and related to the Thurston polytope for 3-manifolds. Applications of it to BNS-invariants have been established by Kielak.

- There is the conjecture that for an aspherical closed manifold with vanishing simplicial volume in the sense of Gromov & Thurston all its L^2-invariants vanish.